International Journal of Theoretical Physics, Vol. 36, No. 12, 1997

Invariant Star-Product on a Poisson–Lie Group and h-Deformation of the Corresponding Lie Algebra

M. Mansour[1]

Received June 25, 1997

In this paper we prove that a left-invariant star-product on a Poisson–Lie group leads to the quantum Lie algebra structure on the corresponding Lie algebra of the Lie group.

1. INTRODUCTION

A great deal of attention and effort has recently been devoted in theoretical physics to the mathematical structures referred to as quantum groups (Drinfeld, 1987; Jimbo, 1985, 1989; Manin, 1988).

The interest in quantum groups arise almost simultaneously in statistical mechanics as well as in conformal theories, in solid-state physics as well as in the study of topologically nontrivial solutions of nonlinear equations, so that the research in quantum groups grew along parallel lines from physical as well as mathematical problems.

As it is well known, quantum groups can be seen as a noncommutative generalization of topological spaces which have a group structure; such a structure induces an abelian Hopf algebra (Abe, 1980) structure on the algebra of functions defined on the group.

Quantum groups are defined then as a non-abelian Hopf algebra (Takhtajan, 1989). A way to generate them consists in deforming the product of an abelian Hopf algebra into a non-abelian one (*-product) using the so-called quantization by the deformation procedure or star-quantization (Bayen *et al.*, 1978a, b). This quantization technique gives a deformed product once it is assigned a Poisson bracket on the algebra of functions on the group.

[1]Laboratoire de Physique Theorique, Departement de Physique, FACULTÉ DES SCIENCES, B.P. 1014, Rabat, Morocco.

0020-7748/97/1200-3007$12.50/0 © 1997 Plenum Publishing Corporation

Classical r-matrices (Drinfeld, 1983a; Semenov-Tian-Shansky, 1985; Lu and Weinstein, 1990) play a very important role in the theory of quantum groups. They are closely related to the structure of a Poisson–Lie group which appears as the classical limit of a quantum group.

The present paper is organized as follows. In Section 2 a brief review is given on the relation between an invariant star-product and the quantum Yang–Baxter equation (QYBE). Section 3 gives the basic definitions of quantum Lie algebra and quantum Killing form. Section 4 shows that the invariant star-product on a Poisson–Lie group G leads to the structure of a quantum Lie algebra on the corresponding Lie algebra of the Lie group. Finally, Section 5 deals with the case of $sl(2)$ in an explicit way.

2. THE STAR-PRODUCT AND QYBE

Let Λ^l be the left-invariant Poisson structure on a Lie group **G** defined by the element $r \in (\mathrm{g} \wedge \mathrm{g})$ (Moreno and Valero, 1992) (g is the Lie algebra of **G**), which satisfies the classical Yang–Baxter equation:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0 \tag{2.1}$$

i.e.,

$$\Lambda^l(g) = T_e L_g r, \qquad \forall g \in G \tag{2.2}$$

where L_g is the left translation of **G**, and $T_e L_g$ is the tangent map of L_g in e (unit element of **G**).

A left-invariant star-product on **G** is a deformation of the associative algebra $C^\infty(\mathbf{G})$ of C^∞-functions on **G** with respect to the usual product, defined as

$$\varphi *^l \psi = \varphi \cdot \psi + h\{\varphi, \psi\}^l + \sum_{n=2}^{\infty} h^n C_n(\varphi, \psi) \tag{2.3}$$

where $\varphi . \psi$ is the usual multiplication of functions $\varphi, \psi \in C^\infty(\mathbf{G})$, $\{\cdot, \cdot\}^l$ is the Poisson bracket corresponding to the 2-tensor Λ^l, C_i is a left-invariant bidifferential operator on **G** such that

$$\text{(i)} \qquad C_i(1, \varphi) = C_i(\varphi, 1) = 0 \qquad \forall \varphi \in C^\infty(\mathbf{G}) \tag{2.4}$$

$$\text{(ii)} \qquad C_i(\varphi, \psi) = (-1)^i C_i(\psi, \varphi) \qquad \forall \varphi, \psi \in C^\infty(\mathbf{G}) \tag{2.5}$$

and $\forall \varphi, \psi, \chi \in C^\infty(\mathbf{G})$ we have

$$\varphi *^l (\psi *^l \chi) = (\varphi *^l \psi) *^l \chi \tag{2.6}$$

Moreno and Valero (1992) and Drinfeld (1983b) prove that the C_i (φ, ψ) is given by

$$C_i(\varphi, \psi) = (F_i(x, y))^l(\varphi \otimes \psi) \tag{2.7}$$

where $(F_i\ (x, y))^l$ is the left-invariant bidifferential operator determined by $F_i(x, y) \in \mathscr{U}(\mathscr{g}) \otimes \mathscr{U}(\mathscr{g})$.

Thus, the $*^l$-product (left invariant star-product) becomes the element

$$F(x, y) = 1 + \sum_{n=1}^{\infty} F_n(x, y)h^n$$

in $\mathscr{U}(\mathscr{g}) \otimes \mathscr{U}(\mathscr{g})[[h]]$ such that

$$F(x + y, z)F(x, y) = F(x, y + z)F(y, z) \tag{2.8}$$

Then, the element (Drinfeld, 1983b)

$$S(x, y) = F^{-1}(y, x)F(x, y) \tag{2.9}$$

satisfies

$$S(x, y)S(x, z)S(y, z) = S(y, z)S(x, z)S(x, y) \tag{2.10}$$

$$S(x, y)S(y, x) = 1 \tag{2.11}$$

i.e., $S(x, y)$ satisfies the triangular QYBE.

3. DEFORMATION OF THE LIE ALGEBRA

We recall that a Lie algebra g is a vector space endowed with a bilinear map

$$[\ ,\ ]\colon\ \mathscr{g} \otimes \mathscr{g} \to \mathscr{g}$$

such that

$$(1) \qquad [\ ,\ ] = -[\ ,\ ] \circ \boldsymbol{\tau} \tag{3.1}$$

$$(2) \qquad [\ ,\ ]([\ ,\ ] \otimes \mathrm{id})(\mathrm{id} + \boldsymbol{\tau}^{12}\boldsymbol{\tau}^{23} + \boldsymbol{\tau}^{23}\boldsymbol{\tau}^{12}) = 0. \tag{3.2}$$

Furthermore, the Killing form on g is a bilinear map

$$\langle\ ,\ \rangle\colon\ \mathscr{g} \otimes \mathscr{g} \to \mathbf{K}$$

such that

$$(\mathrm{i}) \qquad \langle\ ,\ \rangle = \langle\ ,\ \rangle \circ \boldsymbol{\tau} \tag{3.3}$$

$$(\mathrm{ii}) \qquad \langle\ ,\ \rangle([\ ,\ ] \otimes \mathrm{id})(\mathrm{id} + \boldsymbol{\tau}^{23}) = 0 \tag{3.4}$$

where τ is the transposition operator

$$\tau\colon \ \mathscr{g} \otimes \mathscr{g} \to \mathscr{g} \otimes \mathscr{g}$$
$$(a \otimes b) \mapsto b \otimes a \tag{3.5}$$

and

$$\tau^{23} = 1 \otimes \tau, \qquad \tau^{12} = \tau \otimes 1 \in \mathrm{End}(\mathscr{g} \otimes \mathscr{g} \otimes \mathscr{g}). \tag{3.6}$$

Here we define a structure of the quantum Lie algebra on g called the S-Lie algebra (Gurevich and Rubstov, 1990; Gurevich, 1990) as a bilinear map from $\mathscr{g} \otimes \mathscr{g}$ to $\mathscr{g}[[h]]$ given by

$$[\ ,\ ]_s\colon \ \mathscr{g} \otimes \mathscr{g} \to \mathscr{g}[[h]]$$

such that

$$(1) \qquad [\ ,\ ]_s = -[\ ,\ ]_s \circ S \tag{3.7}$$

$$(2) \qquad [\ ,\ ]_s([\ ,\ ]_s \otimes \mathrm{id})(\mathrm{id} + S^{12}S^{23} + S^{23}S^{12}) = 0 \tag{3.8}$$

where

$$S = \tau + \sum_{i \geq 1} h^i S_i, \qquad S_i \in \mathrm{End}(\mathscr{g} \otimes \mathscr{g}) \tag{3.9}$$

satisfies the quantum Yang–Baxter equation

$$S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23} \tag{3.10}$$

and reduces to τ when $h = 0$.

The quantum Killing form on $\mathscr{g}$ is defined as a bilinear map

$$\langle\ ,\ \rangle_s\colon \ \mathscr{g} \otimes \mathscr{g} \to \mathbf{K}[[h]]$$

such that

$$(1) \qquad \langle\ ,\ \rangle_s = \langle\ ,\ \rangle_s \circ S \tag{3.11}$$

$$(2) \qquad \langle\ ,\ \rangle_s([\ ,\ ] \otimes \mathrm{id})(\mathrm{id} + S^{23}) = 0 \tag{3.12}$$

where S in (3.11) and (3.12) is the same as the S given in (3.9).

4. INVARIANT STAR-PRODUCT AND DEFORMATION OF THE LIE ALGEBRA

Let $F(x, y) \in \mathscr{U}(\mathscr{g}) \otimes \mathscr{U}(\mathscr{g})[[h]]$ be a $*^l$ product on G, this implies that

$$F(x + y, z)F(x, y) = F(x, y + z)F(y, z) \tag{4.1}$$

$$F = 1 + \frac{h}{2} r + \theta(h^2) \tag{4.2}$$

where r is a solution of the triangular classical Yang–Baxter equation, i.e., $r \in \mathscr{g} \wedge \mathscr{g} \subset \mathscr{g} \otimes \mathscr{g}$ and satisfies the following equation:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0 \tag{4.3}$$

where $r^{12} = r \otimes \mathbf{1} \in \mathscr{U}(\mathscr{g})^{\otimes 3}$, and so on. $\mathscr{U}(\mathscr{g})$ denotes the universal enveloping algebra.

The inverse of F is given by

$$F^{-1} = 1 - \frac{h}{2} r + \hat{\theta}(h^2). \tag{4.4}$$

Let ρ be the adjoint representation of $\mathscr{g}$ in End($\mathscr{g}$) and define

$$\check{F} = (\rho \otimes \rho)F \in \mathrm{End}(\mathscr{g} \otimes \mathscr{g})[[h]] \tag{4.5}$$

given explicitly by

$$\check{F} = \mathbf{1} \otimes \mathbf{1} + \frac{h}{2} R + \tilde{\theta}(h^2) \tag{4.6}$$

where

$$R = (\rho \otimes \rho)r, \qquad \tilde{\theta}(h^2) = (\rho \otimes \rho)\theta(h^2). \tag{4.7}$$

In terms of a basis $\{x_i\}$ of $\mathscr{g}$, write $r = r^{ij} x_i \otimes x_j$ (summation under repeated indices is understood) and for $x, y \in \mathscr{g}$, we have

$$\check{F}(x \otimes y) = x \otimes y + \frac{h}{2} r^{ij}[x_i, x] \otimes [x_j, y] + \tilde{\theta}(x, y, h^2) \tag{4.8}$$

the inverse $\check{F}^{-1}$ is given by

$$\check{F}^{-1}(x \otimes y) = x \otimes y - \frac{h}{2} r^{ij}[x_i, x] \otimes [x_j, y] + \bar{\theta}(x, y, h^2) \tag{4.9}$$

We also define

$$\tilde{S} = (\rho \otimes \rho)(Z \circ S) = (\rho \otimes \rho)(F^{-1} \cdot F_{21}) = \check{F}^{-1} \circ Z \circ F \tag{4.10}$$

such that for any $x, y \in \mathscr{g}$, we have at the first order

$$\tilde{S}(x, y) = y \otimes x - hr^{ij}[x_i, y] \otimes [x_j, x] + \Theta(x, y, h^2). \tag{4.11}$$

Then by an easy calculus we can show that $\tilde{S}$ satisfies the quantum Yang–Baxter equation

$$\tilde{S}_{12}\tilde{S}_{23}\tilde{S}_{12} = \tilde{S}_{23}\tilde{S}_{12}\tilde{S}_{23}. \tag{4.12}$$

Indeed at the first order we obtain that

$$\tilde{S}_{12}\tilde{S}_{23}\tilde{S}_{12}(x \otimes y \otimes z) = z \otimes y \otimes x - hr^{ij}([x_i, z] \otimes [x_j, y] \otimes x + [x_i, z] \otimes y \otimes [x_j, x] + z \otimes [x_i, y] \otimes [x_j, x]) + \theta(x, y, h^2) \tag{4.13}$$

and

$$\tilde{S}_{23}\tilde{S}_{12}\tilde{S}_{23}(x \otimes y \otimes z) = z \otimes y \otimes x - hr^{ij}(z \otimes [x_i, y] \otimes [x_j, x] + [x_i, z] \otimes y \otimes [x_j, x] + [x_i, z] \otimes [x_j, y] \otimes x) + \theta(x, y, h^2). \tag{4.14}$$

Now using the $\tilde{S}$-matrix, we can define a quantum Lie algebra structure on the corresponding Lie algebra $\mathscr{g}$. Then if we introduce $[\ ,\ ]_{\tilde{s}}$ as

$$[\ ,\ ]_{\tilde{s}} = [\ ,\ ] \circ \check{F} \tag{4.15}$$

where $[\ ,\ ]$ is the classical one, we have for any $x, y \in \mathscr{g}$

$$[\ ,\ ]_{\tilde{s}}(x \otimes y) = [\ ,\ ] \circ \check{F}(x \otimes y) = [x, y] + \frac{h}{2} r^{ij} \Big[[x_i, x], [x_j, y] \Big] \bmod h^2. \tag{4.16}$$

First, using the fact that $\tilde{S}^2 = \mathrm{id}$, we obtain that

$$[\ ,\ ]_{\tilde{s}} = -[\ ,\ ]_{\tilde{s}} \circ \tilde{S}. \tag{4.17}$$

Second, using the fact that $[\ ,\ ]$ satisfies the classical Jacobi identity and that r is a solution of the classical Yang–Baxter equation, we can show that $[\ ,\ ]_{\tilde{s}}$, satisfies the following quantum Jacobi identity:

$$[\ ,\ ]_{\tilde{s}}([\ ,\ ]_{\tilde{s}} \otimes \mathrm{id})(\mathrm{id} + \tilde{S}^{12}\tilde{S}^{23} + \tilde{S}^{23}\tilde{S}^{12}) = 0. \tag{4.18}$$

If we define the quantum Killing form as

$$\langle\ ,\ \rangle_{\tilde{s}} = \langle\ ,\ \rangle \circ \check{F} \tag{4.19}$$

where $\langle\ ,\ \rangle$ is the classical one, such that for any $x, y \in \mathscr{g}$ that we obtain,

$$\langle x, y\rangle_{\tilde{s}} = \langle x, y\rangle + \frac{h}{2} r^{ij} \langle [x_i, x], [x_j, y]\rangle + \theta(h^2) \tag{4.20}$$

then we can easily show that

$$\langle\ ,\ \rangle_{\tilde{s}} = \langle\ ,\ \rangle_{\tilde{s}} \circ \tilde{S}. \tag{4.21}$$

Using the fact that $\langle\ ,\ \rangle$ is $\mathrm{ad}_{\mathscr{g}}$-invariant, we can prove that

$$\langle\ ,\ \rangle_{\tilde{s}}([\ ,\ ] \otimes \mathrm{id})(\mathrm{id} \otimes \tilde{S}^{23}) = 0. \tag{4.22}$$

5. STAR-PRODUCT ON $SL(2,C)$ AND h-DEFORMATION OF THE $sl(2)$

Let $G = SL(2, C)$; the corresponding Lie algebra $sl(2)$ is generated by X, Y, H such that

$$[X, Y] = H \tag{5.1}$$

$$[X, H] = 2X \tag{5.2}$$

$$[Y, H] = -2Y. \tag{5.3}$$

A solution of the classical Yang–Baxter equation is

$$r = H \wedge X \in \Lambda^2(sl(2)). \tag{5.4}$$

An invariant star-product on $SL(2, C)$ is given by (Ohn, 1992)

$$F = \exp\left[\frac{1}{2}\,\Delta H - \frac{1}{2}\left(H\,\frac{\sinh(hX)}{hX} \otimes e^{-hX} + e^{hX} \otimes H\,\frac{\sinh(hX)}{hX}\right)\frac{h\Delta X}{\sinh(h\Delta X)}\right]$$

where Δ is the usual comultiplication on the enveloping algebra $U(sl(2))$. This implies that:

$$\check{F} = \exp\left[\frac{1}{2}\,(\check{H} \otimes 1 + 1 \otimes \check{H}) - \frac{1}{2}\left(\check{H}\,\frac{\sinh(h\check{X})}{h\check{X}} \otimes e^{-h\check{X}} + e^{h\check{X}} \otimes \check{H}\,\frac{\sinh(h\check{X})}{h\check{X}}\right)\frac{h\Delta\check{X}}{\sinh(h\Delta\check{X})}\right]$$

where

$$\check{H} = \rho(H), \qquad \check{X} = \rho(X), \qquad \check{Y} = \rho(Y). \tag{5.5}$$

Then, the quantum Lie algebra structure on $sl(2)$ is given by

$$\begin{aligned}[X, H]_{\tilde{s}} &= [\ ,\ ] \circ \check{F}(X \otimes H)\\ &= (2X) \mod h^2\end{aligned} \tag{5.6}$$

$$\begin{aligned}[Y, H]_{\tilde{s}} &= [\ ,\ ] \circ \check{F}(Y \otimes H)\\ &= (2Y - 2hH) \mod h^2\end{aligned} \tag{5.7}$$

$$\begin{aligned}[X, Y]_{\tilde{s}} &= [\ ,\ ] \circ \check{F}(X \otimes Y)\\ &= (H - 2hX) \mod h^2.\end{aligned} \tag{5.8}$$

The quantum Killing form on $sl(2)$ is as follows:

$$\langle X, Y\rangle_{\tilde{s}} = (\langle X, Y\rangle - h\langle X, H\rangle) \mod h^2 \tag{5.9}$$

$$\langle H, X\rangle_{\tilde{s}} = (\langle H, X\rangle + 2h\langle X, X\rangle) \mod h^2 \tag{5.10}$$

$$\langle H, Y\rangle_{\tilde{s}} = (\langle H, Y\rangle + 2h\langle X, Y\rangle) \mod h^2 \tag{5.11}$$

6. CONCLUSION

An h-deformation of a bialgebra as an algebra is given by an invariant star-product on the associated connected and simply connected Poisson–Lie group.

ACKNOWLEDGMENT

I would like to thank Prof. F. Guédira for helpfull discussions.

REFERENCES

Abe, E. (1980). *Hopf Algebras*, Cambridge University Press, Cambridge.

Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D. (1978a). Deformation theory and quantization, I, *Annals of Physics*, **111**, 61.

Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D. (1978b). Deformation theory and quantization, II, *Annals of Physics*, **110**, 111.

Drinfeld, V. G. (1983a). *Soviet Mathematics Doklady*, **27**, 86.

Drinfeld, V. G. (1983b). On constant quasiclassical solution of the QYBE, *Soviet Mathematics Doklady*, **28**, 667–671.

Drinfeld, V. G. (1987). Quantum groups in *Proceedings International Conference of Mathematicians, Berkeley, 1986*, AMS, Providence, Rhode Island.

Gurevich, D. (1990). Equation de Yang–Baxter et quantification des cocycles, *Comptes Rendus de l'Academic des Sciences Paris*, **1310**, 845–848.

Gurevich, D. and Rubstov, V. (1990). Yang–Baxter and deformation of associative and Lie algebras algebras, in *Lectures Notes in Mathematics*, Vol. 1510, Springer, Berlin, p. 47.

Jimbo, M. (1985). Quantum R-matrix related to the toda system, an algebraic approach, in *Lecture Notes in Physics*, Vol. 246, Springer, Berlin, pp. 335–360.

Jimbo, M. (1989). *International Journal of Modern Physics, A*, **4**, 3759.

Lu, J.-H. and Weinstein, A. (1990). *Journal of Differential Geometry*, **31**, 501.

Manin, Yu. I. (1988). *Quantum Groups and Non-commutative Geometry*, Centre de Recherches Mathématiques, Montreal, Canada.

Moreno, C., and Valero, L. (1992). Star-products and quantization of Poisson–Lie groups, *Journal of Geometry, and Physics*, **9**, 369–402.

Ohn, C. (1992). *Lectures in Mathematical Physics*, Vol. 25, Springer, Berlin, pp. 85–88.

Semenov-Tian-Shansky, M. (1985). *Publications RIMS, Kyoto*, **21**, 1237.

Takhtajan, L. (1989). *Introduction to Quantum Groups and Integrable Models of Quantum Field Theories*, Lecture Notes in Physics, Vol. 370, Springer, Berlin.